Intro Video: Section 2.5 continuity

Math F251X: Calculus 1

What is continuity? What does it mean to say a function is continuous?

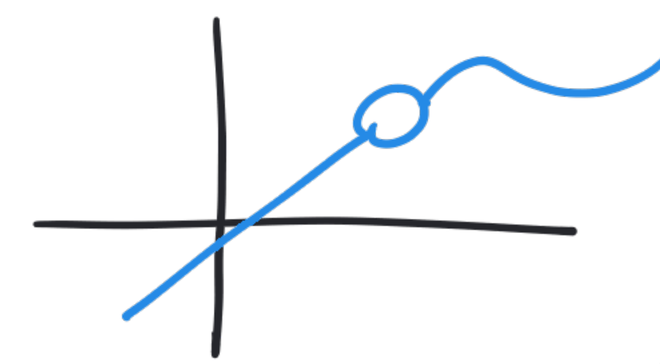
Intuitively:

No holes

No jumps

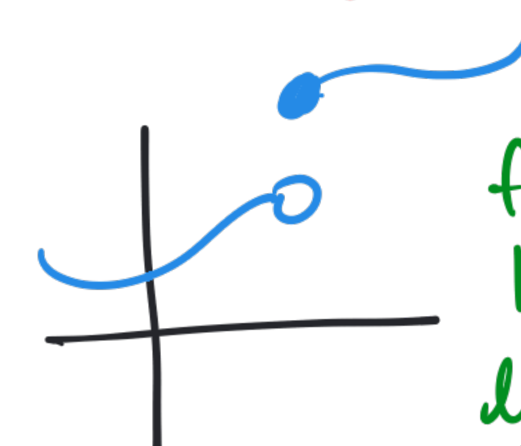
No asymptotes

Not like this!



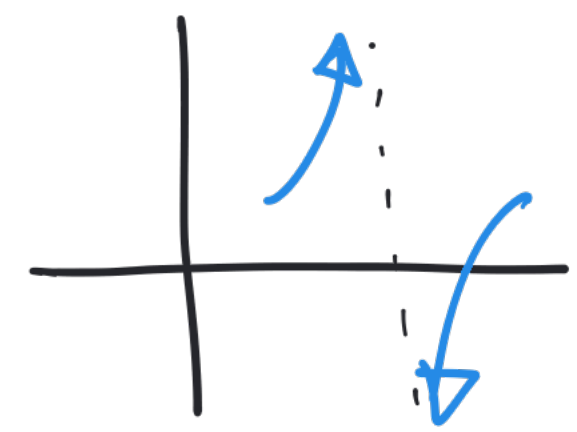
limit exists, f(a) DNE

or this



f(a) exists but $\lim_{x \to a} f(x)$ DNE

or this!

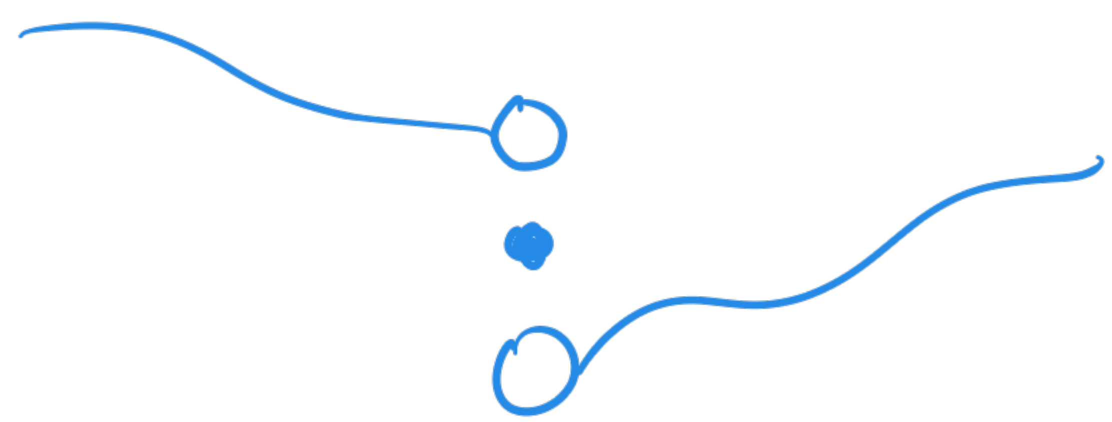


<u>Definition</u>: A function f is <u>continuous</u> <u>at a</u> $\iff \lim_{x \to a} f(x) = f(a)$.

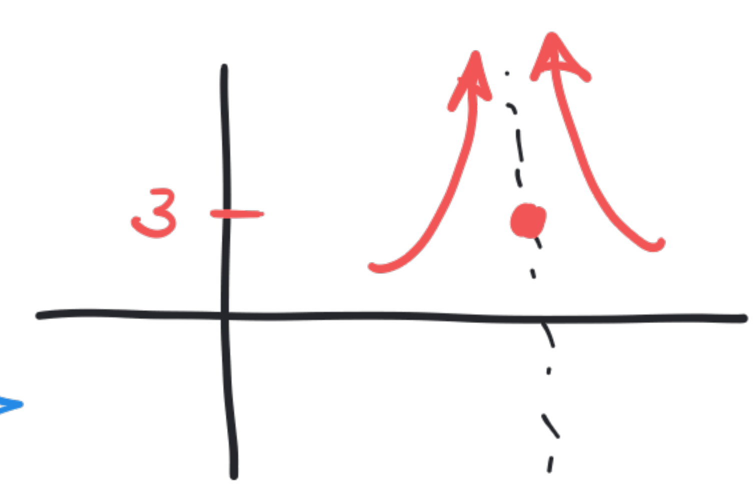
Continuous from the left: $\lim_{x \to a^-} f(x) = f(a)$ 

Continuous from the right: $\lim_{x \to a^+} f(x) = f(a)$ 

Can have an example where both limits exist but still not continuous



WARNING "the limit exists" means limit is finite!

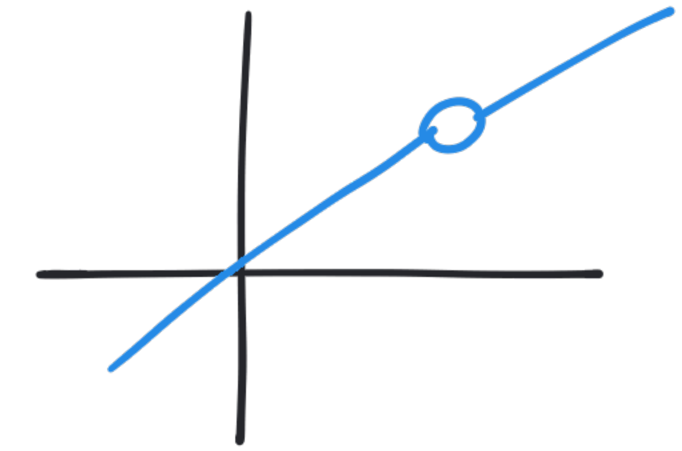


Not Continuous!

$f(a) = 3,\quad \lim_{x \to a} f(x) = \infty$

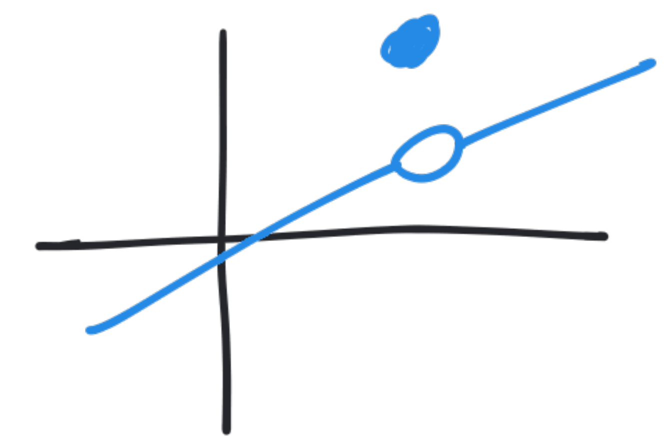
Classification of discontinuities

Holes/removable discontinuity



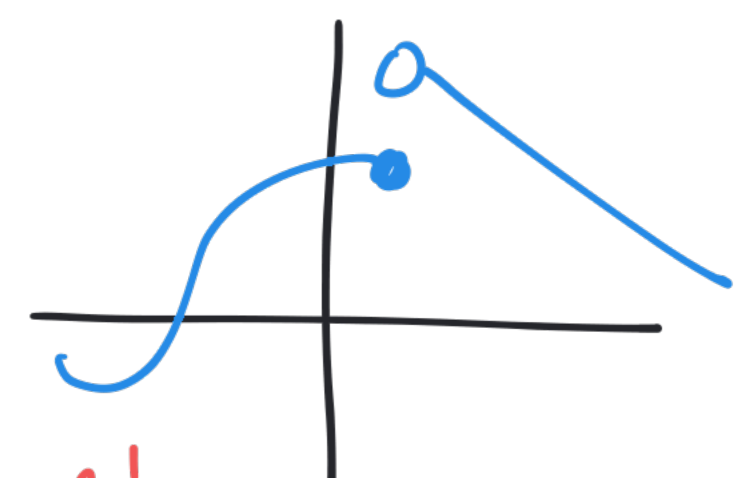
$\lim_{x \to a} f(x)$ but doesn't equal $f(a)$

(including because $f(a)$ DNE)



Jump discontinuity

$\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists, but they're not equal!

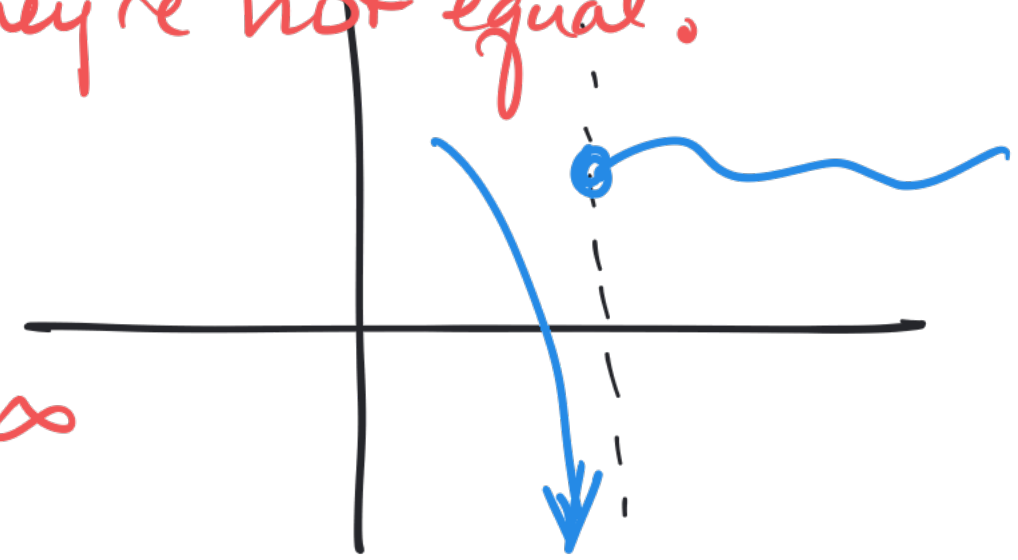


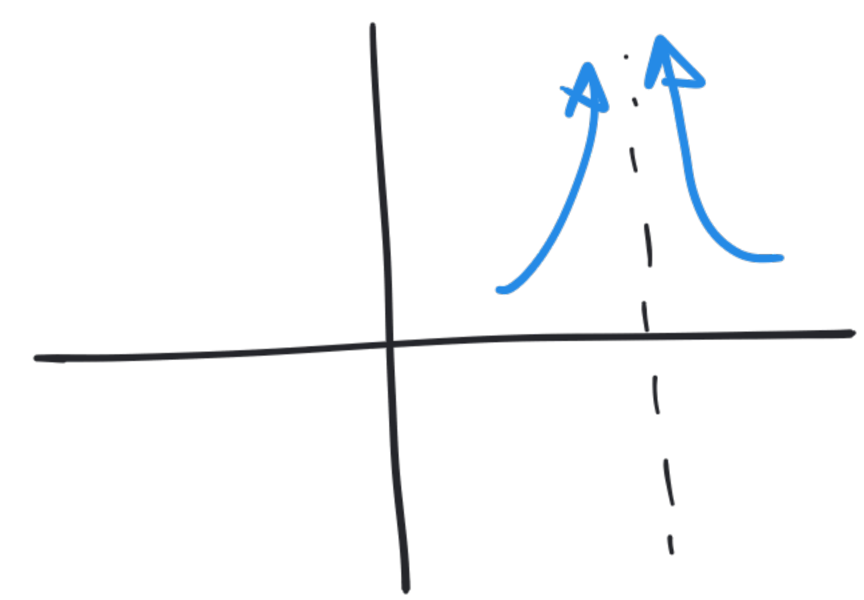
Asymptotes

$\lim_{x \to a^-} f(x) = +\infty$ or $-\infty$

or

$\lim_{x \to a^+} f(x) = +\infty$ or $-\infty$.





Example: Consider $f(x) = \begin{cases} \boxed{\dfrac{(x+3)(x-4)}{x+3}} & \text{if } x < 1 \\ x^2 & \text{if } 1 \leq x < 2 \\ 2+x & \text{if } x > 2 \\ 4 & \text{if } x = 2 \end{cases}$

No other gaps!

At what points in its domain is $f(x)$ continuous?

Where it is not continuous, what kind of discontinuity is there?

DOMAIN $= (-\infty, -3) \cup (-3, \infty)$

Investigate: -3, 1, 2

Use definition of continuity!

Is it true that $\lim_{x \to a} f(x) = f(a)$?

If not, classify!

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{(x+3)(x-4)}{(x+3)}$$

$$= \lim_{x \to -3} x - 4 = -3 - 4 = 7$$

$\lim_{x \to -3} f(x) \neq f(-3)$ because -3 is not in the domain!

REMOVABLE!

Example: Consider $f(x) = \begin{cases} \boxed{\dfrac{(x+3)(x-4)}{x+3}} & \text{if } x<1 \\ x^2 & \text{if } 1 \le x < 2 \\ 2+x & \text{if } x>2 \\ 4 & \text{if } x=2 \end{cases}$

- $x=-3$ is a removable discontinuity because $\lim_{x\to -3} f(x) = 7$ but $f(-3)$ DNE.

② $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} \frac{(x+3)(x-4)}{(x+3)} = \lim_{x\to 1} x-4 = -3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1^2 = 1$$

$$f(1) = 1^2 = 1$$

Continuous from right

JUMP DISCONTINUITY

③ $x=2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2+x = 2+2 = 4$$

$$f(2) = 4$$

Continuous at $x=2$!

Our favorite functions are continuous!

① Polynomials are continuous on $(-\infty, \infty)$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_i \text{ constants}$$

② Rational functions (quotients of polynomials)

Continuous <u>on their domain</u>

<u>Ex</u> $f(x) = \dfrac{x+5}{x-2}$ Domain is $(-\infty, 2) \cup (2, \infty)$

Continuous everywhere <u>except</u> 2, which is not in the domain

③ $f(x) = e^x$ ④ $g(x) = \ln(x)$ (on $(0, \infty)$)

⑤ trig functions (on their domains)

⑥ $f(x) = \sqrt[n]{x}$ ⑦ Inverse trig functions

$\boxed{\text{Combining continuous functions}}$

If $f(x)$ and $g(x)$ are both continuous, then

$f(x) = \sin(x)$ $\quad$ $g(x) = 3x^2$

$f(x) + g(x)$ is continuous $\quad$ $\sin(x) + 3x^2$ is conts!

$f(x)g(x)$ is continuous $\quad$ $3x^2 \sin(x)$

$c\,f(x)$ is continuous $\quad$ $45 \sin(x)$

$\dfrac{f(x)}{g(x)}$ is continuous on its domain $\quad$ $\dfrac{\sin(x)}{3x^2}$ is conts on $(-\infty, 0) \cup (0, \infty)$

What about composition?

FACT (theorem): If f is continuous, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Example: What is $\displaystyle\lim_{x \to 1} \ln\left(\frac{5-x^2}{1+x}\right)$?

$\ln(x)$ is continuous on $(0,\infty)$ and $1 \in (0,\infty)$

So $\displaystyle\lim_{x \to 1} \ln\left(\frac{5-x^2}{1+x}\right) = \ln\left(\lim_{x \to 1} \frac{5-x^2}{1+x}\right)$

in the domain of $\dfrac{5-x^2}{1+x}$

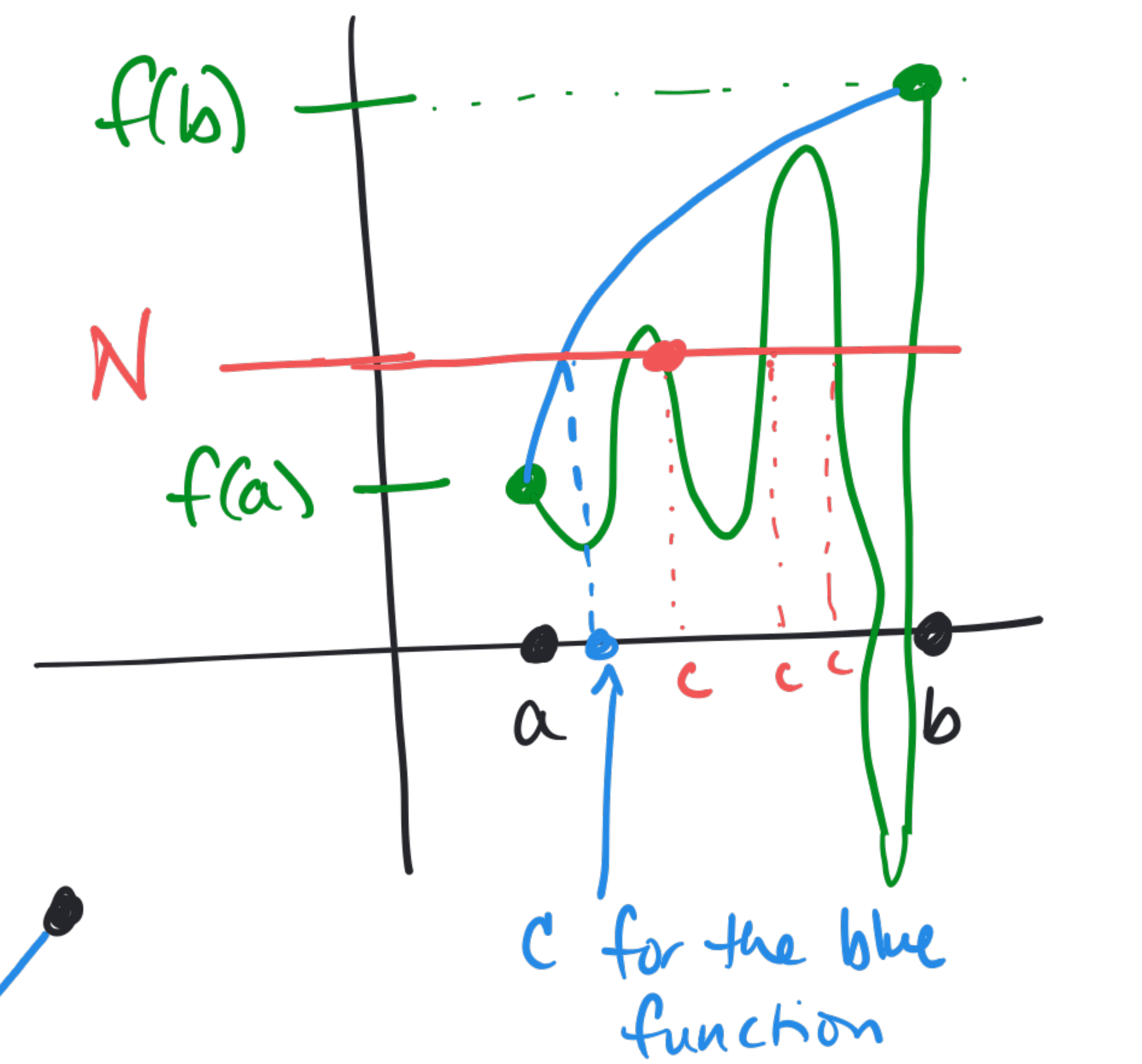
$$= \ln\left(\frac{5-1^2}{1+1}\right) = \ln\left(\frac{4}{2}\right) = \ln(2)$$

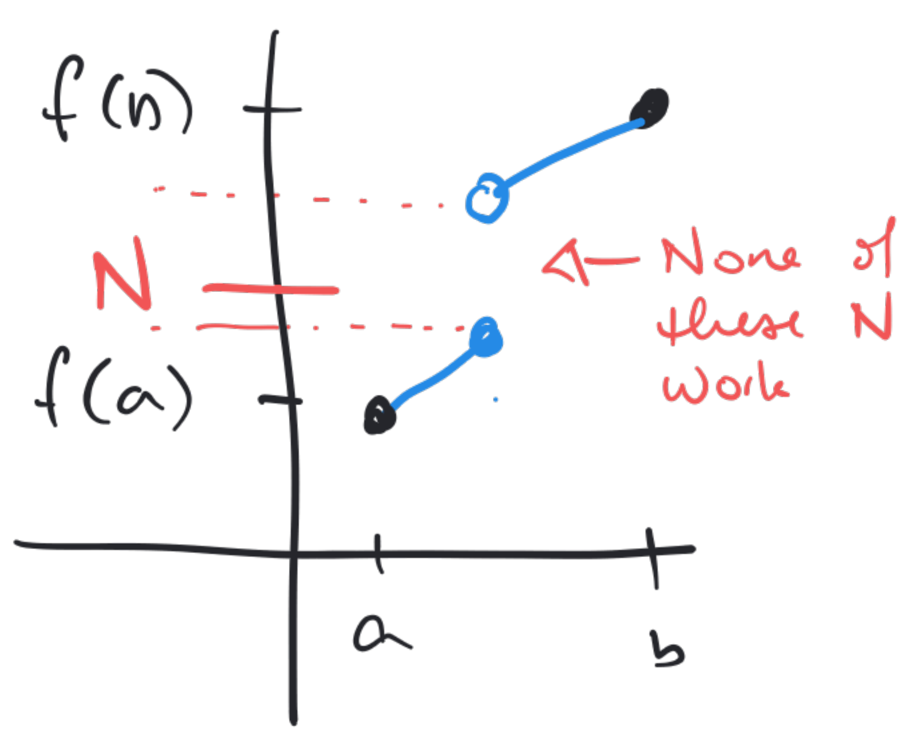
Intermediate Value Theorem

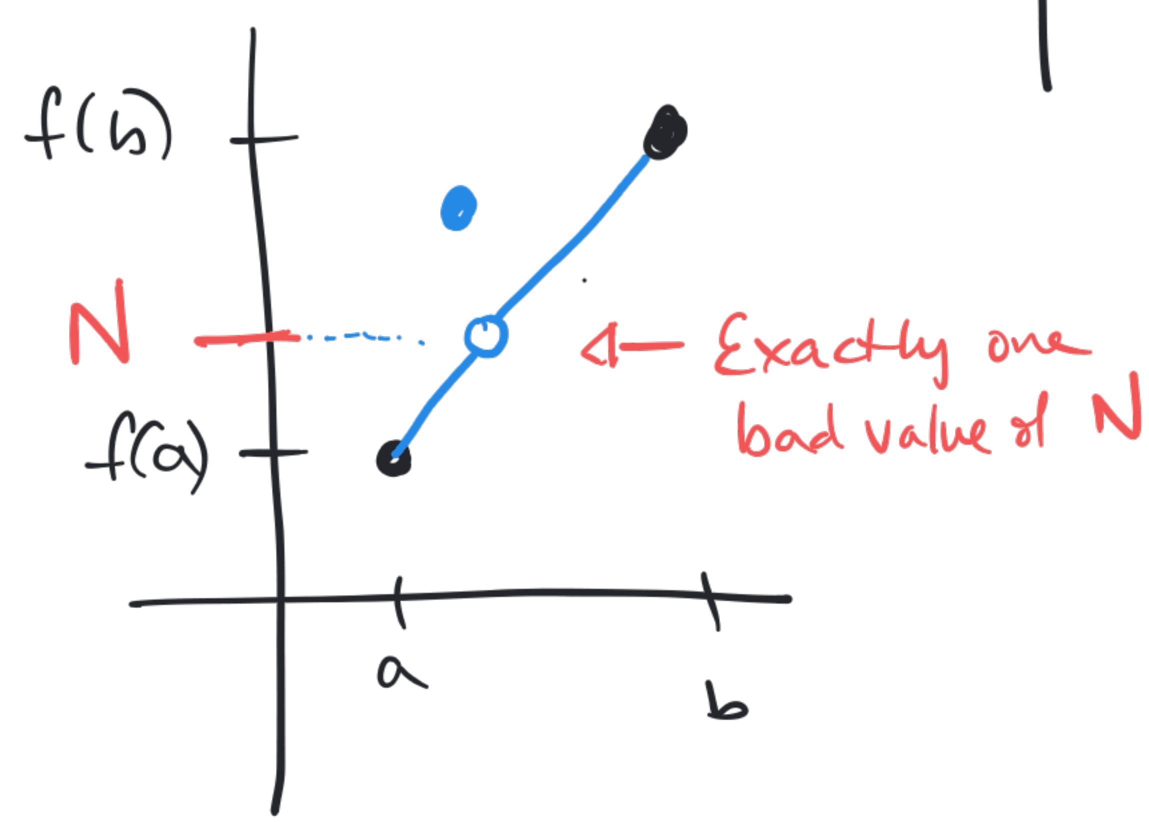
Suppose $f(x)$ is continuous on $[a, b]$.

Let N be any number between $f(a)$ and $f(b)$.

Then there must exist $c \in (a,b)$ so that $f(c) = N$







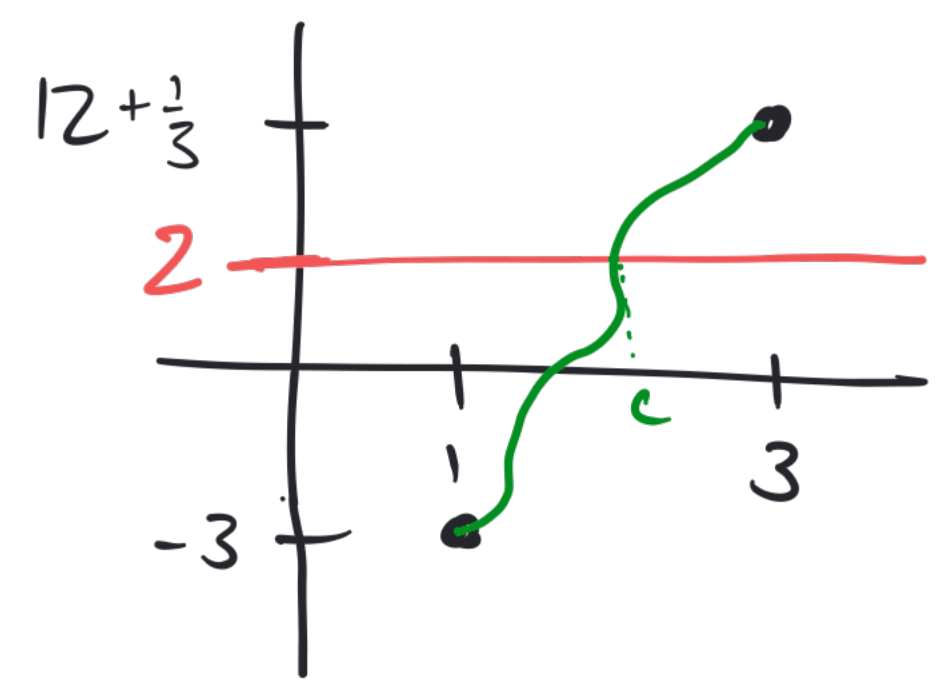
Example: Let $f(x) = x^3 - 5x + \frac{1}{x}$.

Show that there exists a solution to $f(x) = 2$ in the interval $(1, 3)$.

$$f(1) = 1^3 - 5(1) + \frac{1}{1} = 1 - 5 + 1 = -3$$

$$f(3) = 3^3 - 5(3) + \frac{1}{3} = 27 - 15 + \frac{1}{3} = 12 + \frac{1}{3}$$

Observe $2 \in \underbrace{\left(f(1), f(3) \right)}_{\text{interval}} = (-3, 12\tfrac{1}{3})$



By the intermediate value theorem there must exist some $c \in (-3, 12\tfrac{1}{3})$ so that $f(c) = 2$. This c is our solution!